Almost even-Clifford hermitian manifolds with large automorphism group

Gerardo Arizmendi*†, Rafael Herrera‡§ and Noemi Santana¶‖

Abstract

We study manifolds endowed with an (almost) even Clifford (hermitian) structure and admitting a large automorphism group. We classify them when they are simply connected and the dimension of the automorphism group is maximal, and also prove a gap theorem for the dimension of the automorphism group.

1 Introduction

Recently, there has been some interest in manifolds admitting so-called even Clifford structures [9, 10]. Here, we study such manifolds when they admit a large automorphism group. This type of problem has been studied on Riemannian manifolds [13, 14], almost hermitian manifolds [12], and almost quaternion-hermitian manifolds [11].

It is a classical result [6] that the maximal dimension of the isometry group of a connected n-dimensional Riemannian manifold is $\frac{1}{2}n(n+1)$. If the dimension is maximal, the manifold is isometric to either Euclidean space $\mathbb{R}^n$, or the sphere S^n, or projective space $\mathbb{RP}^n$, or (simply connected) hyperbolic space. Furthermore, in [13] it was shown that the isometry group contains no m-dimensional closed subgroup where

$$\frac{1}{2}n(n-1)+1 < m < \frac{1}{2}n(n+1).$$

In [12], it was shown that the automorphism group of a connected $2n$-dimensional almost-hermitian manifold has dimension at most $n(n+2)$. If the dimension of the automorphism is maximal, the manifold is isometric to either complex Euclidean space $\mathbb{C}^n$, or an open ball with Kähler structure with negative constant holomorphic sectional curvature, or complex projective space $\mathbb{CP}^n$. In this case, however, there is also a (unique) manifold whose automorphism group has dimension one less than the maximal one, namely, euclidean space [4].

In [11], it was shown that the automorphism group of a connected $4n$-dimensional almost quaternion-hermitian manifold has dimension at most $2n^2+5n+3$. If the dimension of the automorphism group is between $2n^2+5n$ and $2n^2+5n+3$, the manifold is isometric to either quaternionic Euclidean space $\mathbb{H}^n$, or quaternionic projective space $\mathbb{HP}^n$, or quaternionic hyperbolic space.

In this paper, we will prove the analogous theorems for almost even-Clifford hermitian manifolds of rank $r \geq 3$. Our terminology differs from that of [9, 10] since we have added the words "almost" and "hermitian" since, in principle, there is no integrability condition on the structure and the compatibility

*Centro de Investigación en Matemáticas, A. P. 402, Guanajuato, Gto., C.P. 36000, México. E-mail: gerardo@cimat.mx
†Partially supported by a CONACYT scholarship
‡Centro de Investigación en Matemáticas, A. P. 402, Guanajuato, Gto., C.P. 36000, México. E-mail: rherrera@cimat.mx
§Partially supported by grants from CONACyT and LAISLA (CONACyT-CNRS)
¶Instituto de Matemáticas, UNAM, Unidad Cuernavaca, A.P. 6–60, C.P. 62131, Cuernavaca, Morelos, México. E-mail: noemi.santana@im.unam.mx
‖Partially supported by grants of CONACyT and LAISLA (CONACyT-CNRS).

with a Riemannian metric is an extra condition. We shall explore integrability conditions in the style of Gray [3] in a future paper.

The note is organized as follows. In Section 2 we recall some preliminaries on Clifford algebras and representations, and almost even-Clifford hermitian manifolds. In Section 3, we give and upper bound in the dimension of the automorphism group (Proposition 3.1), determine the spaces whose automorphism group has dimension equal to this bound (Theorem 3.1), and prove a gap in the dimension of the automorphism group (Proposition 3.2).

Acknowledgments. The second named author would like to thank the International Centre for Theoretical Physics and the Institut des Hautes Études Scientifiques for their hospitality and support.

2 Preliminaries

In this section we recall material that can also be consulted in [2, 7]. Let Cl_r denote the Clifford algebra generated by all the products of the orthonormal vectors $e_1, e_2, \ldots, e_r \in \mathbb{R}^r$ subject to the relations

$$e_j e_k + e_k e_j = -2\delta_{jk}, \quad \text{for } 1 \le j, k \le r.$$

The even Clifford subalgebra Cl_r^0 is defined as the invariant $(+1)$-subspace of the involution of Cl_r induced by the map $-\mathrm{Id}_{\mathbb{R}^r}$. The Spin group $Spin(r) \subset Cl_r$ is the subset

$$Spin(r) = \{x_1 x_2 \cdots x_{2l-1} x_{2l} \mid x_j \in \mathbb{R}^r,\ |x_j| = 1,\ l \in \mathbb{N}\},$$

endowed with the product of the Clifford algebra. The Lie algebra of $Spin(r)$ is

$$\mathfrak{spin}(r) = \mathrm{span}\{e_i e_j \mid 1 \le i < j \le r\}.$$

Now, we summarize some results about real representations of Cl_r^0 in the next table (cf. [7]). Here d_r denotes the dimension of an irreducible representation of Cl_r^0 and v_r the number of distinct irreducible representations. Let $\tilde{\Delta}_r$ denote the irreducible representation of Cl_r^0 for $r \not\equiv 0 \pmod 4$ and $\tilde{\Delta}_r^{\pm}$ denote the irreducible representations for $r \equiv 0 \pmod 4$.

$r \pmod 8$	d_r	Cl_r^0	$\tilde{\Delta}_r$ / $\tilde{\Delta}_r^{\pm} \cong \mathbb{R}^{d_r}$	v_r
1	$2^{\lfloor \frac{r}{2} \rfloor}$	$\mathbb{R}(d_r)$	$\mathbb{R}^{d_r}$	1
2	$2^{\frac{r}{2}}$	$\mathbb{C}(d_r/2)$	$\mathbb{C}^{d_r/2}$	1
3	$2^{\lfloor \frac{r}{2} \rfloor + 1}$	$\mathbb{H}(d_r/4)$	$\mathbb{H}^{d_r/4}$	1
4	$2^{\frac{r}{2}}$	$\mathbb{H}(d_r/4) \oplus \mathbb{H}(d_r/4)$	$\mathbb{H}^{d_r/4}$	2
5	$2^{\lfloor \frac{r}{2} \rfloor + 1}$	$\mathbb{H}(d_r/4)$	$\mathbb{H}^{d_r/4}$	1
6	$2^{\frac{r}{2}}$	$\mathbb{C}(d_r/2)$	$\mathbb{C}^{d_r/2}$	1
7	$2^{\lfloor \frac{r}{2} \rfloor}$	$\mathbb{R}(d_r)$	$\mathbb{R}^{d_r}$	1
8	$2^{\frac{r}{2} - 1}$	$\mathbb{R}(d_r) \oplus \mathbb{R}(d_r)$	$\mathbb{R}^{d_r}$	2

Table 1

Note that the representations are complex for $r \equiv 2, 6 \pmod 8$ and quaternionic for $r \equiv 3, 4, 5 \pmod 8$.

Almost even-Clifford hermitian structures

Definition 2.1 *A* linear even-Clifford hermitian structure of rank r *on* $\mathbb{R}^N$, $N \in \mathbb{N}$, *is a representation*

$$Cl_r^0 \longrightarrow \mathrm{End}(\mathbb{R}^N)$$

such that each bivector $e_i e_j$, $1 \le i < j \le r$, *is mapped to an antisymmetric endomorphism* J_{ij} *satisfying*

$$J_{ij}^2 = -\mathrm{Id}_{\mathbb{R}^N}. \tag{1}$$

Notice that the subalgebra $\mathfrak{spin}(r)$ is mapped injectively into the skew-symmetric endomorphisms $\mathrm{End}^-(\mathbb{R}^N)$.

First, let us assume $r \not\equiv 0 \pmod 4$, $r > 1$. In this case, $\mathbb{R}^N$ decomposes into a sum of irreducible representations of Cl_r^0. Since this algebra is simple, such irreducible representations can only be trivial or copies of the standard representation $\tilde{\Delta}_r$ of Cl_r^0 (cf. [7]). Due to (1), there are no trivial summands in such a decomposition so that

$$\mathbb{R}^N = \tilde{\Delta}_r \otimes_{\mathbb{R}} \mathbb{R}^m$$

for some $m \in \mathbb{N}$. Thus, we see that $\mathfrak{spin}(r)$ has an isomorphic image

$$\widehat{\mathfrak{spin}(r)} := \mathfrak{spin}(r) \otimes \{\mathrm{Id}_{m\times m}\} \subset \mathfrak{so}(d_r m).$$

Secondly, let us assume $r \equiv 0 \pmod 4$. Recall that if $\hat{\Delta}_r$ is the irreducible representation of Cl_r, then by restricting this representation to Cl_r^0 it splits as the sum of two inequivalent irreducible representations

$$\hat{\Delta}_r = \tilde{\Delta}_r^+ \oplus \tilde{\Delta}_r^-.$$

Since $\mathbb{R}^N$ is a representation of Cl_r^0 satisfying (1), there are no trivial summands in such a decomposition so that

$$\mathbb{R}^N \quad = \quad \tilde{\Delta}_r^+ \otimes \mathbb{R}^{m_1} \oplus \tilde{\Delta}_r^- \otimes \mathbb{R}^{m_2},$$

for some $m_1, m_2 \in \mathbb{N}$. By restricting this representation to $\mathfrak{spin}(r) \subset Cl_r^0$, consider

$$\widehat{\mathfrak{spin}(r)} := \mathfrak{spin}(r)^+ \otimes \{\mathrm{Id}_{m_1\times m_1} \oplus \mathbf{0}_{m_2\times m_2}\} \oplus \mathfrak{spin}(r)^- \otimes \{\mathbf{0}_{m_1\times m_1} \oplus \mathrm{Id}_{m_2\times m_2}\} \subset \mathfrak{so}(d_r m_1 + d_r m_2),$$

where $\mathfrak{spin}(r)^\pm$ are the images of $\mathfrak{spin}(r)$ in $\mathrm{End}(\tilde{\Delta}_r^\pm)$ respectively.

Definition 2.2
- *A* rank r almost even-Clifford hermitian structure, $r \geq 2$, *on a Riemannian manifold M is a smoothly varying choice of linear even-Clifford hermitian structure on each tangent space of M. Let $Q \subset \mathrm{End}^-(TM)$ denote the subbundle with fiber $\mathfrak{spin}(r)$.*
- *A Riemannian manifold carrying such a structure will be called an* almost even-Clifford hermitian manifold.
- *An almost even-Clifford hermitian structure on a Riemannian manifold M is called* parallel *if the bundle Q is parallel with respect to the Levi-Civita connection on M.*

Notice that the definition of (parallel) even-Clifford structure in [9] implies the one we have just given.

3 Automorphism group

In this section we derive an upper bound on the dimension of the automorphism group of an almost even-Clifford hermitian manifold and classify the manifolds whose automorphism group's dimension attains such an upper bound.

The *automorphism group* of an almost even-Clifford hermitian manifold M, denoted by $\mathrm{Aut}(M)$, is the (sub)group of isometries which preserve the almost even-Clifford hermitian structure. A vector field X on M is an *infinitesimal automorphism* if it is a Killing vector field that preserves the structure, i.e. locally

$$\mathcal{L}_X J_{ij} = \sum_{k<l} \alpha_{kl}^{(ij)} J_{kl},$$

for some (local) functions $\alpha_{kl}^{(ij)}$, where $\mathcal{L}_X$ denotes the Lie derivative in the direction of X. These vector fields form the Lie algebra $\mathrm{aut}(M)$ of $\mathrm{Aut}(M)$.

3.1 Upper bound

Let X be an infinitesimal automorphism of M. Consider

$$\mathcal{L}_X(J_{ij}(Y)) \quad = \quad (\mathcal{L}_X J_{ij})(Y) + J_{ij}(\mathcal{L}_X Y),$$

i.e.

$$\nabla_X(J_{ij}(Y)) - \nabla_{J_{ij}(Y)}X \quad = \quad \sum_{k<l} \alpha_{kl}^{(ij)}\, J_{kl}(Y) + J_{ij}(\nabla_X Y - \nabla_Y X).$$

Now suppose we are calculations at a point p where $X_p = 0$, so that

$$-\nabla_{J_{ij}(Y)}X \quad = \quad \sum_{k<l} \alpha_{kl}^{(ij)}\, J_{kl}(Y) - J_{ij}(\nabla_Y X),$$

i.e.

$$[J_{ij}, \nabla X](Y) \quad = \quad \sum_{k<l} \alpha_{kl}^{(ij)}\, J_{kl}(Y).$$

Hence, $(\nabla X)_p$ is a skew-symmetric endomorphism such that

$$[J_{ij}, \nabla X] \quad = \quad \sum_{k<l} \alpha_{kl}^{(ij)}\, J_{kl}.$$

i.e. $(\nabla X)_p$ belongs to the normalizer of $\widehat{\mathfrak{spin}(r)} = \mathrm{span}(J_{ij})$ in $\mathrm{End}^-(T_pM) = \mathfrak{so}(T_pM)$. Such a normalizer has been calculated in [1] and we list them for $r \geq 3$.

r (mod 8)	N	$N_{\mathfrak{so}(N)}(\widehat{\mathfrak{spin}(r)})$	$C_{\mathfrak{so}(N)}(\widehat{\mathfrak{spin}(r)})$
0	$d_r(m_1+m_2)$	$\mathfrak{so}(m_1)\oplus\mathfrak{so}(m_2)\oplus\mathfrak{spin}(r)$	$\mathfrak{so}(m_1)\oplus\mathfrak{so}(m_2)$
1	d_rm	$\mathfrak{so}(m)\oplus\mathfrak{spin}(r)$	$\mathfrak{so}(m)$
2	d_rm	$\mathfrak{u}(m)\oplus\mathfrak{spin}(r)$	$\mathfrak{u}(m)$
3	d_rm	$\mathfrak{sp}(m)\oplus\mathfrak{spin}(r)$	$\mathfrak{sp}(m)$
4	$d_r(m_1+m_2)$	$\mathfrak{sp}(m_1)\oplus\mathfrak{sp}(m_2)\oplus\mathfrak{spin}(r)$	$\mathfrak{sp}(m_1)\oplus\mathfrak{sp}(m_2)$
5	d_rm	$\mathfrak{sp}(m)\oplus\mathfrak{spin}(r)$	$\mathfrak{sp}(m)$
6	d_rm	$\mathfrak{u}(m)\oplus\mathfrak{spin}(r)$	$\mathfrak{u}(m)$
7	d_rm	$\mathfrak{so}(m)\oplus\mathfrak{spin}(r)$	$\mathfrak{so}(m)$

Table 2

Proposition 3.1 *Let M be a N-dimensional almost even-Clifford hermitian manifold. Then*

r (mod 8)	$N=\dim(M)$	*upper bound* $d_{max} \geq \dim(\mathrm{Aut}(M))$
0	$d_r(m_1+m_2)$	$\binom{m_1}{2}+\binom{m_2}{2}+\binom{r}{2}+d_r(m_1+m_2)$
$1,7$	d_rm	$\binom{m}{2}+\binom{r}{2}+d_rm$
$2,6$	d_rm	$m^2+\binom{r}{2}+d_rm$
$3,5$	d_rm	$\binom{2m+1}{2}+\binom{r}{2}+d_rm$
4	$d_r(m_1+m_2)$	$\binom{2m_1+1}{2}+\binom{2m_2+1}{2}+\binom{r}{2}+d_r(m_1+m_2)$

Table 3

□

3.2 Large automorphism group

In this subsection, we determine the spaces that support an automorphism group of maximal dimension and prove a gap in the dimension of the automorphism group.

Proposition 3.2 *Let M be a N-dimensional, rank $r \geq 3$ almost even-Clifford hermitian manifold and assume that the dimension of its automorphism group is maximal. Then, for any $p \in M$, the isotropy subgroup A_p of p is conjugate to $C_{SO(N)}(Spin(r)) \cdot Spin(r) \subset SO(N)$.*

Proof. The dimension of the orbit f p under $\mathrm{Aut}(M)$ satisfies

$$\dim(\mathrm{Aut}(M)) - \dim(A_p) \leq N,$$

so that

$$\begin{array}{rcl} \dim(A_p) & \geq & \dim(\mathrm{Aut}(M)) - N \\ & = & d_{max} - N \\ & = & \dim(C_{SO(N)}(Spin(r)) \cdot Spin(r)). \end{array}$$

The Lie algebra $\mathfrak{a}_p$ of A_p maps one-to-one into $C_{\mathfrak{so}(N)}(\widehat{\mathfrak{spin}(r)}) \oplus \widehat{\mathfrak{spin}(r)}$ since a Killing vector field X is determined by its values X_p and $(\nabla X)_p$. Hence,

$$\mathfrak{a}_p \cong C_{\mathfrak{so}(N)}(\widehat{\mathfrak{spin}(r)}) \oplus \widehat{\mathfrak{spin}(r)}.$$

□

Proposition 3.3 *Let M be a rank $r \geq 3$ almost even-Clifford hermitian manifold and assume that the dimension of its automorphism group is maximal. Then, M is symmetric and the almost even-Clifford hermitian structure is parallel.*

Proof. Let $p \in M$ and A_p denote its isotropy group. We know that $A_p = C_{SO(N)}(Spin(r)) \cdot Spin(r)$. Since $C_{SO(N)}(Spin(r))$ contains 1 and $Spin(r)$ contains -1, we have $-1 \in A_p$. Thus, there is an element $g \in A_p$ whose derivative $dg_p = -\mathrm{Id}_{T_pM}$ in the isotropy representation of A_p on T_pM. In other words, the automorphism g is a (global) symmetry at p and M is symmetric. Since these symmetries generate the translations along geodesics, M has a transitive group of automorphisms, not just isometries.

Proceeding as in [5, p. 264], given a vector field W with $W_p \neq 0$, let $c(t)$ be the geodesic with $\dot{c}(0) = W_p$ and τ_t be the group of translations along c. Then

$$Z_q := \frac{d}{dt}\tau_t(q)_{|t=0}$$

is an infitesimal automorphism since τ_t are automorphisms. We have that $W_p = Z_p$. For $v \in T_pM$, let $\gamma(s)$ be a curve with $\gamma'(0) = v$. Then

$$\begin{array}{rcl} \nabla_v Z_p & = & \nabla_{\frac{\partial}{\partial s}} \dfrac{\partial}{\partial t}\tau_t(\gamma(s))_{|s=t=0} \\ & = & \nabla_{\frac{\partial}{\partial t}} \dfrac{\partial}{\partial s}\tau_t(\gamma(s))_{|s=t=0} \\ & = & \nabla_{\frac{\partial}{\partial t}} D\tau_t(v)_{|t=0} \\ & = & 0, \end{array}$$

since D_{τ_t} is parallel transport along c and $D_{\tau_t}(v)$ is a parallel vector field along c. Hence, for any vector $W_p \in T_pM$, we have an infinitesimal isometry Z such that

$$Z_p = W_p \qquad \text{and} \qquad (\nabla Z)_p = 0.$$

Now, recall that

$$\nabla_Z(Y) = \mathcal{L}_Z Y + \nabla_Y Z.$$

On the one hand,

$$\begin{aligned}(\nabla_W(J_{ij}(Y)))_p &= (\nabla_Z(J_{ij}(Y)))_p \\ &= ((\nabla_Z J_{ij})(Y))_p + (J_{ij}(\nabla_Z Y))_p,\end{aligned}$$

and, on the other,

$$\begin{aligned}(\mathcal{L}_Z(J_{ij}(Y)) + \nabla_{J_{ij}(Y)}Z)_p &= ((\mathcal{L}_Z J_{ij})(Y))_p + (J_{ij}(\mathcal{L}_Z Y))_p + (\nabla_{J_{ij}(Y)}Z)_p \\ &= \left(\sum_{k<l}\alpha_{kl}^{(ij)} J_{kl}(Y)\right)_p + (J_{ij}(\nabla_Z Y))_p - (J_{ij}(\nabla_Y Z))_p + (\nabla_{J_{ij}(Y)}Z)_p,\end{aligned}$$

so that

$$\begin{aligned}((\nabla_Z J_{ij})(Y))_p &= \left(\sum_{k<l}\alpha_{kl}^{(ij)} J_{kl}(Y)\right)_p - (J_{ij}(\nabla_Y Z))_p + (\nabla_{J_{ij}(Y)}Z)_p \\ &= \left(\sum_{k<l}\alpha_{kl}^{(ij)} J_{kl}(Y)\right)_p + [(\nabla Z)_p, (J_{ij})_p](Y_p) \\ &= \left(\sum_{k<l}\alpha_{kl}^{(ij)} J_{kl}(Y)\right)_p,\end{aligned}$$

i.e.

$$(\nabla_W J_{ij})_p = \left(\sum_{k<l}\alpha_{kl}^{(ij)} J_{kl}\right)_p .$$

□

Theorem 3.1 *Let M be a simply connected Riemannian almost even-Clifford hermitian manifold of rank $r \geq 3$ such that the dimension of its group of automorphisms is maximal. Then M is isometric to one of the following spaces:*

r	M
arbitrary	$\tilde{\Delta}_r^{\times m}$ *or* $(\tilde{\Delta}_r^+)^{\times m_1} \oplus (\tilde{\Delta}_r^-)^{\times m_2}$, *for some* $m, m_1, m_2 \in \mathbb{N}$
3	$\mathrm{Sp}(k+1)/(\mathrm{Sp}(k)\times\mathrm{Sp}(1)), \mathrm{Sp}(k,1)/(\mathrm{Sp}(k)\times\mathrm{Sp}(1))$
4	$M_1 \times M_2$, *where* $M_i = (\tilde{\Delta}_3)^{\times m}, \mathrm{Sp}(k+1)/(\mathrm{Sp}(k)\times\mathrm{Sp}(1)), \mathrm{Sp}(k,1)/(\mathrm{Sp}(k)\times\mathrm{Sp}(1))$
5	$\mathrm{Sp}(k+2)/(\mathrm{Sp}(k)\times\mathrm{Sp}(2)), \mathrm{Sp}(k,2)/(\mathrm{Sp}(k)\times\mathrm{Sp}(2))$
6	$\mathrm{SU}(k+4)/\mathrm{S}(\mathrm{U}(k)\times\mathrm{U}(4)), \mathrm{SU}(k,4)/\mathrm{S}(\mathrm{U}(k)\times\mathrm{U}(4))$
8	$\mathrm{SO}(k+8)/(\mathrm{SO}(k)\times\mathrm{SO}(8)), SO(k,8)/(\mathrm{SO}(k)\times\mathrm{SO}(8))$
9	$\mathrm{F}_4/\mathrm{Spin}(9), \mathrm{F}_4^{-20}/\mathrm{Spin}(9)$
10	$\mathrm{E}_6/(\mathrm{Spin}(10)\cdot\mathrm{U}(1)), \mathrm{E}_6^{-14}/(\mathrm{Spin}(10)\cdot\mathrm{U}(1))$
12	$\mathrm{E}_7/(\mathrm{Spin}(12)\cdot\mathrm{SU}(2)), \mathrm{E}_7^{-5}/(\mathrm{Spin}(12)\cdot\mathrm{SU}(2))$
16	$\mathrm{E}_8/\mathrm{Spin}^+(16), \mathrm{E}_8^8/\mathrm{Spin}^+(16)$

Table 4

Proof. The flat case is clear and the case $r = 3$ was dealt with in [11],

For $r = 4$, by [9], M is a Riemannian product $M_1 \times M_2$ of quaternion-Kähler manifolds. We claim that

$$\dim(\mathrm{Aut}(M)) = \dim(\mathrm{Aut}(M_1)) + \dim(\mathrm{Aut}(M_2)).$$

Indeed, let $X \in \mathrm{aut}(M)$ an infinitesimal automorphism of M, $X = X_1 + X_2$ with $X_1 \in \Gamma(TM_1)$ and $X_2 \in \Gamma(TM_2)$. We will prove that $X_1 \in \mathrm{aut}(M_1)$ and $X_2 \in \mathrm{aut}(M_2)$. First note that X_1 and X_2 are Killing vector fields.

Recall that

$$\mathcal{L}_X J_{ij} = \sum_{k<l} \alpha_{kl}^{(ij)} J_{kl},$$

and the endomorphisms [9]

$$J_{12}^{\pm} = \pm\frac{1}{2}(J_{14} \pm J_{23}), \qquad J_{31}^{\pm} = \pm\frac{1}{2}(J_{13} \mp J_{24}), \qquad J_{23}^{\pm} = \pm\frac{1}{2}(J_{12} \pm J_{34}),$$

where J_{kl}^{-} and J_{kl}^{+} vanish on M_1 and M_2 respectively. Let $Z = Z_1 + Z_2$ with $Z_1 \in \Gamma(TM_1)$ and $Z_2 \in \Gamma(TM_2)$,

$$\begin{aligned}
(\mathcal{L}_{X_1} J_{ij}^{+})(Z_1)_p &= \mathcal{L}_{X_1}(J_{ij}^{+}(Z_1))_p - J_{ij}^{+}(\mathcal{L}_{X_1} Z_1)_p \in T_pM_1,\\
(\mathcal{L}_{X_1} J_{ij}^{+})(Z_2)_p &= \mathcal{L}_{X_1}(J_{ij}^{+}(Z_2))_p - J_{ij}^{+}(\mathcal{L}_{X_1} Z_2)_p = 0,\\
(\mathcal{L}_{X_2} J_{ij}^{+})(Z_1)_p &= \mathcal{L}_{X_2}(J_{ij}^{+}(Z_1))_p - J_{ij}^{+}(\mathcal{L}_{X_2} Z_1)_p = 0,\\
(\mathcal{L}_{X_2} J_{ij}^{+})(Z_2)_p &= \mathcal{L}_{X_2}(J_{ij}^{+}(Z_2))_p - J_{ij}^{+}(\mathcal{L}_{X_2} Z_2)_p = 0.
\end{aligned}$$

i.e. $\mathcal{L}_X J_{ij}^{+} = \mathcal{L}_{X_1} J_{ij}^{+} \in \mathrm{End}(TM_1)$. Similarly, $\mathcal{L}_X J_{ij}^{-} = \mathcal{L}_{X_2} J_{ij}^{-} \in \mathrm{End}(TM_2)$. Now consider, for instance,

$$\begin{aligned}
\mathcal{L}_X J_{12}^{+} &= \frac{1}{2}\mathcal{L}_X(J_{14} + J_{23})\\
&= \frac{1}{2}\sum_{k<l} \alpha_{kl}^{(14)} J_{kl} + \sum_{k<l} \alpha_{kl}^{(23)} J_{kl}\\
&= \frac{1}{2}\sum \beta_{st} J_{st}^{+} + \sum \gamma_{st} J_{st}^{-},
\end{aligned}$$

for some functions β_{st} and γ_{st}. Since $\mathcal{L}_X J_{12}^{+} \in \mathrm{End}(TM_1)$, all the coefficients $\gamma_{st} = 0$. Therefore

$$\mathcal{L}_{X_1} J_{12}^{+} = \sum \beta_{st} J_{st}^{+},$$

By similar calculations $\mathcal{L}_{X_1} J_{ij}^{+} = \sum \beta_{st}^{(ij)} J_{st}^{+}$ and $\mathcal{L}_{X_2} J_{ij}^{-} = \sum \gamma_{st}^{(ij)} J_{st}^{-}$, i.e. $X_1 \in \mathrm{aut}(M_1)$ and $X_2 \in \mathrm{aut}(M_2)$. Now let $m_i = \dim(M_i)/4$, $i = 1, 2$. Since

$$\begin{aligned}
\dim(\mathrm{Aut}(M)) &= \binom{2m_1 + 1}{2} + \binom{2m_2 + 1}{2} + 6 + 4m_1 + 4m_2\\
&= \dim(\mathrm{Aut}(M_1)) + \dim(\mathrm{Aut}(M_2)),\\
\dim(\mathrm{Aut}(M_i)) &\leq \binom{2m_i + 1}{2} + 3 + 4m_i
\end{aligned}$$

we must have

$$\dim(\mathrm{Aut}(M_i)) = \binom{2m_i + 1}{2} + 3 + 4m_i.$$

Therefore, the dimensions of the automorphism groups of M_1 and M_2 are maximal.

For $r \geq 5$, the symmetric spaces carrying a parallel even Clifford structure were classified in [9] and are listed in Table 4. We claim that the dimension of the automorphism group of each space listed is maximal Indeed, let $M = G/K$ be one of the spaces in Table 4, where G is the group of isometries of M. We need to prove that every Killing vector is an infinitesimal automorphism of M. If $X \in \mathfrak{g}$ is a Killing vector field,

$$\begin{aligned}(\mathcal{L}_X J_{ij})(Z) &= \mathcal{L}_X(J_{ij}(Z)) - J_{ij}(\mathcal{L}_X Z)\\ &= \nabla_X(J_{ij}(Z)) - \nabla_{J_{ij}(Z)}X - J_{ij}(\nabla_X Z) + J_{ij}(\nabla_Z X)\\ &= (\nabla_X J_{ij})(Z) - [\nabla X, J_{ij}](Z)\\ &= \left(\sum_{k<l} a_{kl}^{(ij)} J_{kl}\right)(Z) - [\nabla X, J_{ij}](Z)\end{aligned}$$

since the almost even-Clifford hermitian structure is parallel.

Recall that

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m},$$

and at a point $p \in M$,

$$\mathfrak{m} \cong T_pM,$$

and from the list of possible spaces

$$\mathfrak{k} \cong \mathfrak{a}_p \cong C_{\mathfrak{so}(N)}(\widehat{\mathfrak{spin}(r)}) \oplus \widehat{\mathfrak{spin}(r)}.$$

On the other hand, since M is symmetric,

$$\mathfrak{g} \cong \mathfrak{b}_p \oplus T_pM,$$

so that $\mathfrak{b}_p \cong \mathfrak{k} \cong C_{\mathfrak{so}(N)}(\widehat{\mathfrak{spin}(r)}) \oplus \widehat{\mathfrak{spin}(r)}$ and

$$[\nabla X, J_{ij}] = \sum_{k<l} b_{kl}^{(ij)} J_{kl},$$

i.e.

$$\mathcal{L}_X J_{ij} = \sum_{k<l} (a_{kl}^{(ij)} + b_{kl}^{(ij)}) J_{kl}.$$

□

Theorem 3.2 *Let M be a N-dimensional rank $r \geq 3$, almost even-Clifford hermitian manifold and $p \in M$. Assume the following constraints:*

r (mod 8)	N	*constraint*	*extra constraint*
0	$d_r(m_1+m_2)$	$m_1 \geq m_2 > \binom{r}{2}+1$ or $m_2 \geq m_1 > \binom{r}{2}+1$	$m_1 \equiv m_2 \equiv 0 \pmod 2$
$1,7$	$d_r m$	$m > \binom{r}{2}+1$	$m \equiv 0 \pmod 2$
$2,6$	$d_r m$	$m > \frac{1}{2}\binom{r}{2}+\frac{1}{2}$	$m \equiv 0 \pmod 2$
$3,5$	$d_r m$	$m > \frac{1}{4}\binom{r}{2}+1$	
4	$d_r(m_1+m_2)$	$m_1 \geq m_2 > \frac{1}{4}\binom{r}{2}+1$ or $m_2 \geq m_1 > \frac{1}{4}\binom{r}{2}+1$	

Table 5

If $\dim(\mathrm{Aut}(M))$ *is not maximal, then*

$$\dim(\mathrm{Aut}(M)) < d_{max} - \binom{r}{2}.$$

Proof. Suppose that

$$d_{max} - \binom{r}{2} \leq \dim(\mathrm{Aut}(M)) < d_{max}.$$

At a point $p \in M$, the isotropy group satisfies

$$\begin{aligned} \dim(A_p) &\geq \dim(\mathrm{Aut}(M)) - N \\ &\geq d_{max} - \binom{r}{2} - N \\ &\geq d_C := \dim(C_{SO(N)}(Spin(r))). \end{aligned}$$

The Lie algebra $\mathfrak{a}_p$ of A_p maps one-to-one into $C_{\mathfrak{so}(N)}(\widehat{\mathfrak{spin}(r)}) \oplus \widehat{\mathfrak{spin}(r)}$ since a Killing vector field X is determined by its values X_p and $(\nabla X)_p$. Consider the compositions of this map with the projections to the two factors

$$\rho_1 : \mathfrak{a}_p \longrightarrow C_{\mathfrak{so}(N)}(\widehat{\mathfrak{spin}(r)}), \qquad \rho_2 : \mathfrak{a}_p \longrightarrow \widehat{\mathfrak{spin}(r)}.$$

The subalgebra $\rho_1(\mathfrak{a}_p)$ is either equal to $C_{\mathfrak{so}(N)}(\widehat{\mathfrak{spin}(r)})$ or is contained in a proper maximal subalgebra of $C_{\mathfrak{so}(N)}(\widehat{\mathfrak{spin}(r)})$.

If $r \not\equiv \pm 2 \pmod 8$, the maximal dimension d_M of a proper maximal subalgebra of $C_{\mathfrak{so}(N)}(\widehat{\mathfrak{spin}(r)})$ is given in the following table (see [8]),

$r \pmod 8$	N	d_M	d_C
0	$d_r(m_1+m_2)$	$\max\left\{\binom{m_1-1}{2}+\binom{m_2}{2}, \binom{m_1}{2}+\binom{m_2-1}{2}\right\}$	$\binom{m_1}{2}+\binom{m_2}{2}$
$1,7$	$d_r m$	$\binom{m-1}{2}$	$\binom{m}{2}$
$3,5$	$d_r m$	$\binom{2m-1}{2}+3$	$\binom{2m+1}{2}$
4	$d_r(m_1+m_2)$	$\max\left\{\binom{2m_1-1}{2}+3+\binom{2m_2+1}{2}, \binom{2m_1+1}{2}+\binom{2m_2-1}{2}+3\right\}$	$\binom{2m_1+1}{2}+\binom{2m_2+1}{2}$

Table 6

Thus, due to the constraints on the multiplicities m, m_1, m_2, if $\rho_1(\mathfrak{a}_p)$ is contained in a proper subalgebra of $C_{\mathfrak{so}(N)}(\widehat{\mathfrak{spin}(r)})$,

$$\begin{aligned} d_C &> d_M + \binom{r}{2} \\ &\geq \dim(\rho_1(\mathfrak{a}_p)) + \dim(\rho_2(\mathfrak{a}_p)) \\ &\geq \dim(\mathfrak{a}_p) \\ &\geq d_C, \end{aligned}$$

which is a contradiction. Thus

$$\rho_1(\mathfrak{a}_p) = C_{\mathfrak{so}(N)}(\widehat{\mathfrak{spin}(r)}),$$

and

$$\mathfrak{a}_p \cong C_{\mathfrak{so}(N)}(\widehat{\mathfrak{spin}(r)}) \oplus \mathfrak{K} \subset \mathfrak{so}(N),$$

where

$$\mathfrak{K} := \ker(\rho_1|_{\mathfrak{a}_p}) \subset \ker(\rho_1) = \mathfrak{spin}(r).$$

Therefore $A_p = C_{SO(N)}(Spin(r)) \cdot K$, where K is a closed subgroup of $Spin(r)$. The extra assumptions on m, m_1 and m_2 imply $-1 \in C_{SO(N)}(Spin(r))$ and $-1 \in A_p$. Thus, there is an element $g \in A_p$ whose derivative $dg_p = -\mathrm{Id}_{T_pM}$ in the isotropy representation of A_p on T_pM. In other words, the automorphism g is a (global) symmetry at p and M is symmetric. Since these symmetries generate the translations along geodesics, M has a transitive group of automorphisms, not just isometries. As in the proof of Proposition

3.3, this implies the almost even-Clifford hermitian structure is parallel. By arguments similar to those in the proof of Theorem 3.1, we have $\dim(\mathrm{Aut}(M)) = d_{max}$, which is again a contradiction.

For $r \equiv \pm 2 \pmod 8$, $\dim(A_p) \geq d_C$ will happen if $\mathfrak{su}(m) \subset \rho_1(\mathfrak{a}_p)$. Hence $A_p = H \cdot K$, where H is some subgroup of $C_{SO(N)}(Spin(r))$ containing $SU(m)$ and K is some closed subgroup of $Spin(r)$. Since we are assuming m is even, $-1 \in SU(m)$ and $-1 \in A_p$. Therefore M is symmetric and again, the proofs of Proposition 3.3 and Theorem 3.1 imply $\dim(\mathrm{Aut}(M)) = d_{max}$. □

Remark. The constraints in the previous theorem are given in order to ensure that $-1 \in A_p$. If we were to relax them or change them, a more detailed analysis of the possible subgroups $K \in Spin(r)$ would be needed.

References

[1] Arizmendi, G.; Herrera, R.: Centralizers of spin subalgebras. Preprint (2015)

[2] Friedrich, T.: Dirac operators in Riemannian geometry, AMS.

[3] Gray, A.; Hervella, L.: The sixteen classes of almost hermitian manifolds and their linear invariants. Annali di Matematica Pura ed Applicata 1980, Volume 123, Issue 1, pp 35–58

[4] Ishihara, S.: Groups of isometries of pseudo-Hermitian spaces. I, II. Proc. Japan Acad., **30**, 1954, **31**, 1955.

[5] Jost, J.: Riemannian geometry and geometric analysis. Fifth edition. Universitext. Springer-Verlag, Berlin, 2008. xiv+583 pp. ISBN: 978-3-540-77340-5

[6] Kobayashi, S.; Nomizu, K.: Foundations of differential geometry. Volumen I y II. Wiley, New York, 1963.

[7] Lawson, H. B., Jr.; Michelsohn, M.-L.: Spin geometry. Princeton Mathematical Series, 38. Princeton University Press, Princeton, NJ, 1989. xii+427 pp. ISBN: 0-691-08542-0

[8] Mann, L. N.: Gaps in the dimensions of transformation groups. Illinois J. Math. **10** 1966 532–546.

[9] Moroianu, A.; Semmelmann, U.: Clifford structures on Riemannian manifolds. Adv. Math. **228** (2011), no. 2, 940–967.

[10] Moroianu, A.; Pilca, M.: Higher rank homogeneous Clifford structures J. London Math. Soc. 87, 384-400 (2013).

[11] Santana, N.: Almost quaternion-hermitian manifolds with large automorphism group. J. Geom. Phys. **61** (2011) 347-351.

[12] Tanno, S.: The automorphism groups of almost Hermitian manifolds. Kodai Math. Sem. Rep. Volumen 25, 2, 1973.

[13] Wang, H. C.: Finsler spaces with completely integrable equations of Killing. J. London Math. Soc., 22, 1949.

[14] Wakakuwa, H.: On n-dimensional Riemannian spaces admitting some groups of motions of order less than $\frac{1}{2}n(n-1)$. Tôhoku Math. J. (2), 1954.